Z-NUMBERS—A NEW DIRECTION IN THE ANALYSIS OF UNCERTAIN AND COMPLEX SYSTEMS

Lotfi A. Zadeh

Computer Science Division
Department of EECS
UC Berkeley

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Email: *zadeh@eecs.berkeley.edu*
URL: http://www.cs.berkeley.edu/~zadeh/

INTRODUCTION

BASIC IDEAS

- *Z-number = number + its reliability*
- *Z-number-based models of complex systems—especially economic systems—are more realistic than traditional number-based models.*

PREAMBLE

- *In large measure, science and engineering dwell in the world of measurements and numbers. In this world, a basic question which arises is: How reliable are the numbers which we deal with? This question plays a particularly important role in decision analysis, planning, economics, risk assessment, design and process analysis.*

THE CONCEPT OF A Z-NUMBER (ZADEH 2011)

- ***The concept of a Z-number is intended to provide a basis for computation with numbers which are not totally reliable. More concretely, a Z-number, Z=(A,B), is an ordered pair of two fuzzy numbers. The first number, A, is a restriction on the values which a real-valued variable, X, can take.***

CONTINUED

- *The second number, B, is a restriction on the reliability that X is A. Typically, A and B are described in a natural language.*

Z= (fuzzy value, fuzzy reliability)

$$\begin{array}{cc} \uparrow & \uparrow \\ \textit{NL} & \textit{NL} \end{array}$$

EXAMPLES

- ***A=approximately 2 million dollars***
- ***B=very likely***

- ***A=approximately 1 hour***
- ***B=usually***

- ***A=high***
- ***B=surely***

PRINCIPAL APPLICATIONS OF Z-NUMBERS

- *Of particular importance are applications in economics, decision analysis, risk assessment, prediction, anticipation, planning, biomedicine and rule-based manipulation of imprecise functions and relations.*

THE CONCEPT OF Z-VALUATION

- *A Z-valuation is an ordered triple of the form (X, (A,B)), where X is a real-valued variable and (A,B) is a Z-number.*

Example.

- *(unemployment next year, sharp decline, unlikely)*

CONDITIONAL Z-VALUATION (ZADEH 2011)

If (X, A, B) then (Y, C, D)

Simpler version

If (X, A) then (Y, B, C)

Equivalently,

If (X is A) then (Y iz (B, C))

↑

Z-rule

Z-INFORMATION

- In real-world settings, much of the information in an environment of uncertainty and imprecision may be represented as a collection of Z-valuations and conditional Z-valuations—a collection which is referred to as Z-information.

EXAMPLES OF Z-INFORMATION

- *Usually it is difficult to find parking near the campus in the early morning $\longrightarrow$ (finding parking near the campus in the early morning, difficult, usually)*
- *Usually it takes Robert about an hour to get home from work $\longrightarrow$ (travel time from work to home, about one hour, usually)*

NOTE

- In the analysis of economic systems, much of the information is Z-information.
- Existing economic theories do not reflect this fact.
- Existing models of economic systems are unreliable, E. Derman. “Models Behaving Badly” 2011.

SEMANTICS OF A FUZZY IF-THEN RULE

- ***A concept which plays a pivotal role in uncertain systems analysis is that of a fuzzy if-then rule (Zadeh 1973).***

If X is A then Y is B $\longrightarrow$ ***(X, Y) is A×B***

fuzzy set ***fuzzy set***

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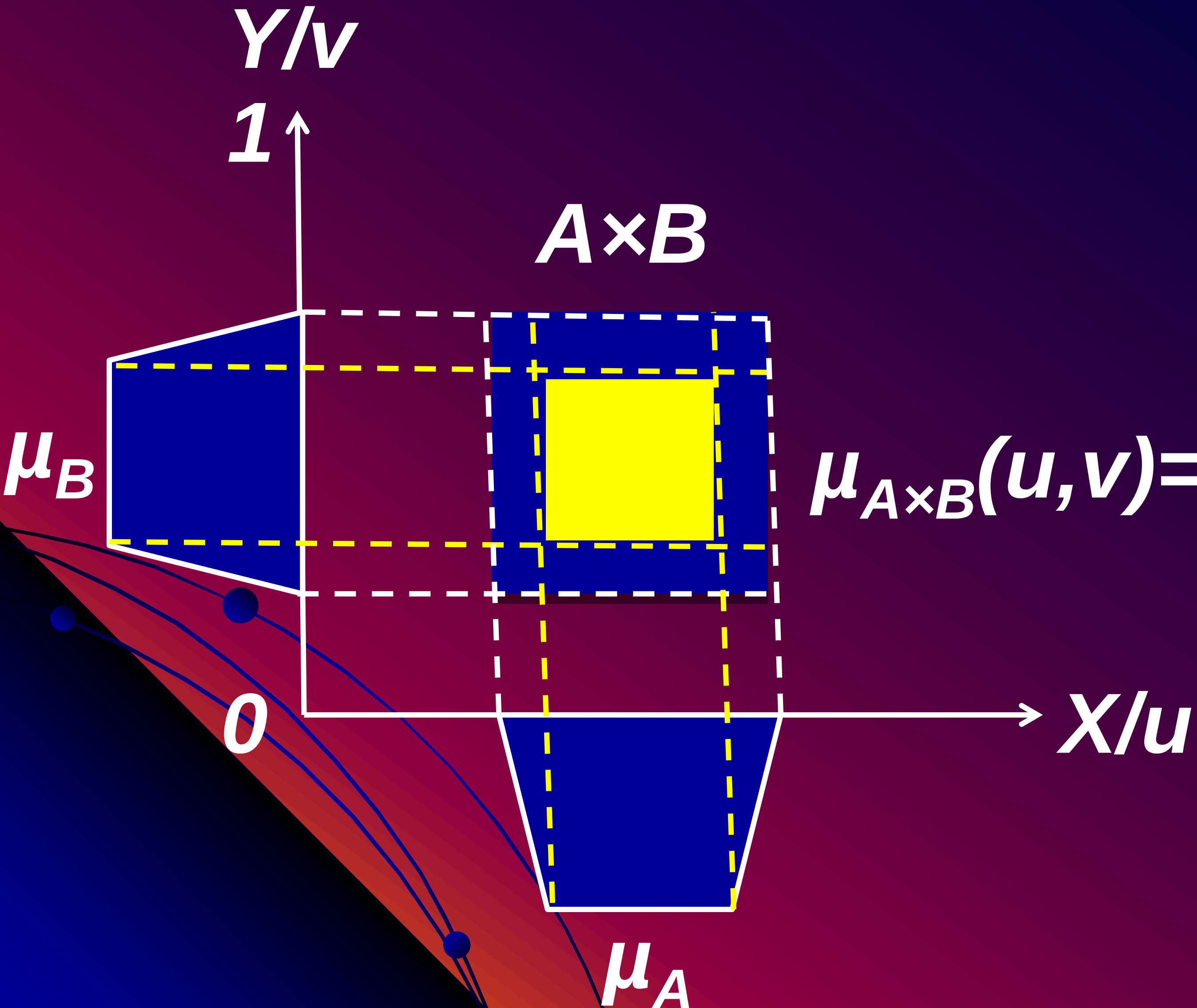


$$\mu_{A\times B}(u,v)=\mu_A(u)\wedge\mu_B(v)$$

FUZZY RULE SETS

- A fuzzy rule set is a collection of fuzzy if-then rules.
- Principally, fuzzy rule sets are used for two purposes: (a) representation of imprecisely defined functions and relations; and (b) approximation to precisely known functions and relations.

BASIC FUZZY RULE SET

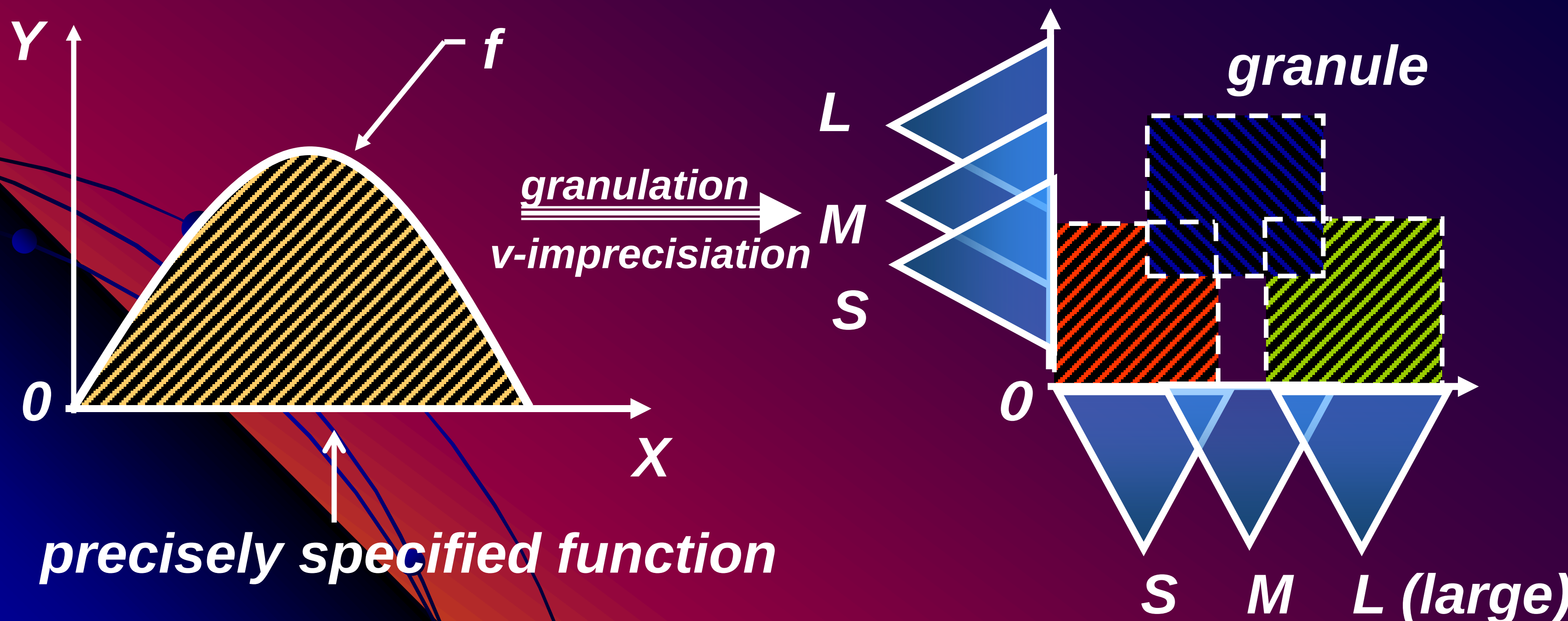
- ***There are many kinds of fuzzy rule sets. A simple fuzzy rule set which is employed in many applications of fuzzy control is expressed as:***

  (If X is A_1 then Y is B_1, … If X is A_n then Y is B_n)

FUZZY LOGIC (FL) GAMBIT

- *Fuzzy logic gambit underlies many important applications of fuzzy logic. In essence,*

  FL gambit = deliberate value imprecisiation (v-imprecisiation) through granulation, followed by meaning precisiation (m-precisiation)



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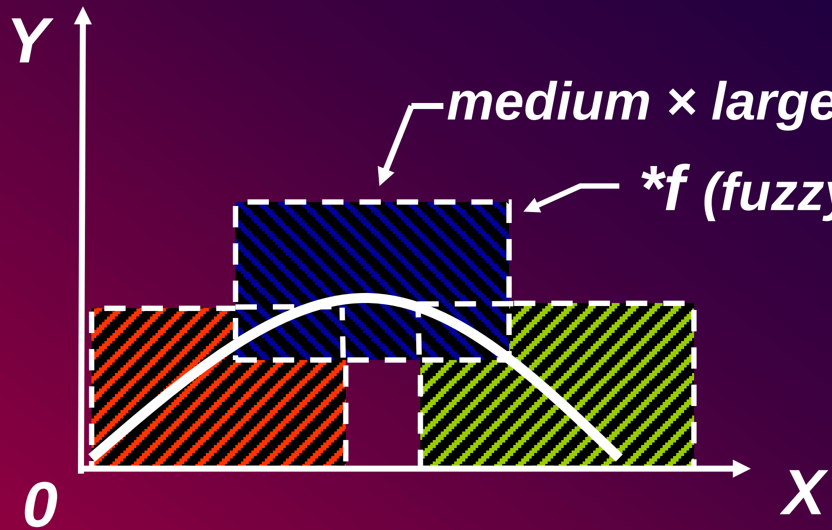


$f \xrightarrow[\text{summarization}]{\text{granulation}}$ *f :

$$*f = \left\{ \begin{array}{l} \text{if } X \text{ is small then } Y \text{ is small} \\ \text{if } X \text{ is medium then } Y \text{ is large} \\ \text{if } X \text{ is large then } Y \text{ is small} \end{array} \right\}$$

$$*f = \left\{ \begin{array}{l} \text{if } X \text{ is small then } Y \text{ is small} \\ \text{if } X \text{ is medium then } Y \text{ is large} \\ \text{if } X \text{ is large then } Y \text{ is small} \end{array} \right\} \xrightarrow{\text{m-precisiation}}$$

small×small+medium×large+large×small

mathematical summary of f

A BASIC PROBLEM (INTERPOLATION) (ZADEH 1976)

If X is A_1 then Y is B_1

…

If X is A_n then Y is B_n

X is A

Y is ?B

- *Y is $m_1 \wedge b_1 + \ldots + m_n \wedge b_n$, where m_i is the degree to which A matches A_i, $i=1, \ldots, n$.*

SEMANTICS OF A Z-RULE

If (X is A) then Y iz (B, C) $\longrightarrow$ (X×Y) iz (A×B, C)

SEMANTICS OF Z-RULE SET / INTERPOLATION

If X is A_1 then Y iz (B_1C_1)

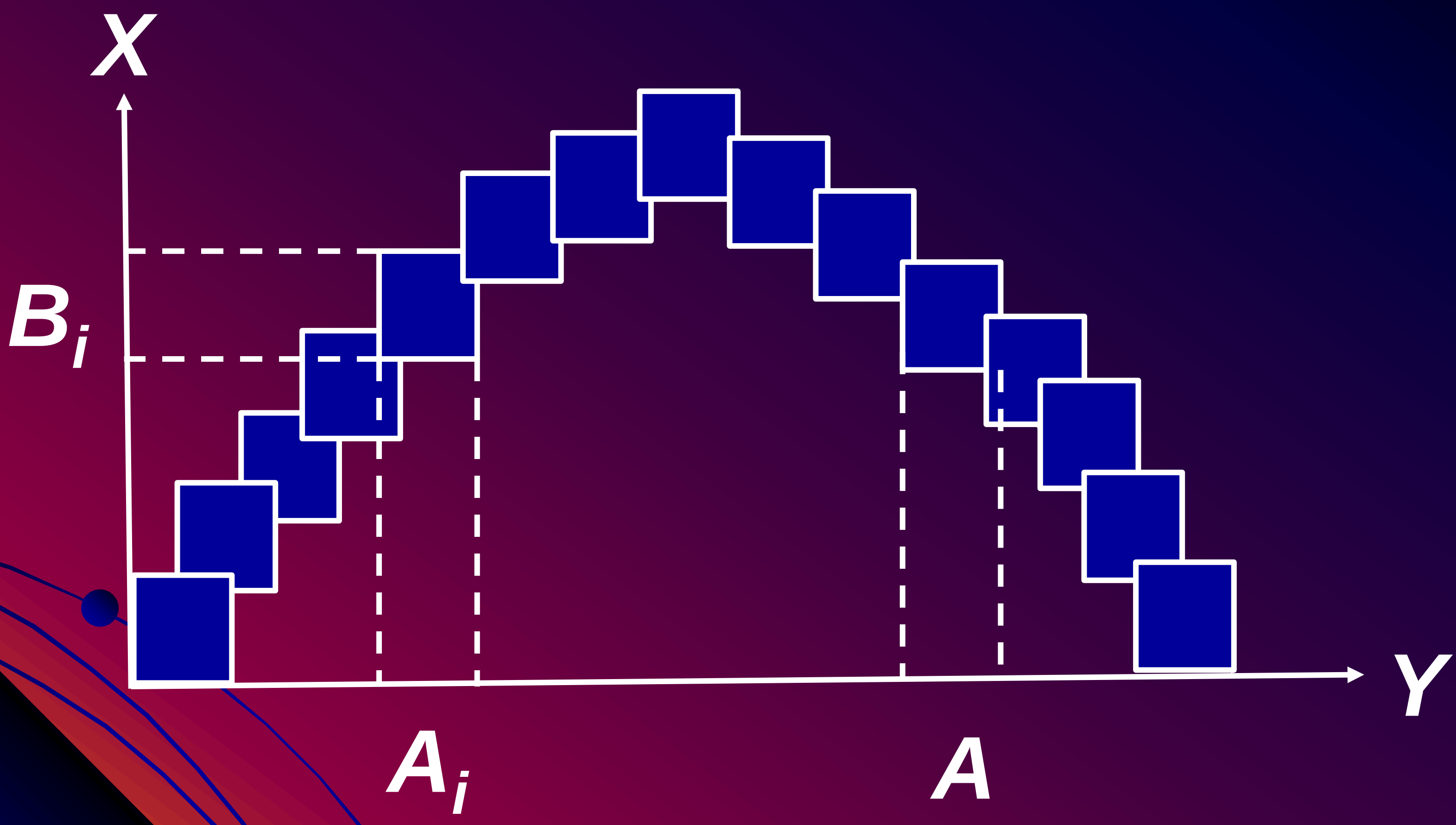
…

If X is A_n then Y iz (B_nC_n) $\longrightarrow$ *?*

(X, Y) iz (($A_1 \times B_1, C_1$) +···+ (X, Y) iz ($A_n \times B_n, C_n$))

- *Interpolation?*

SEMANTICS OF Z-RULE SET / INTERPOLATION



COMPUTATION WITH Z-NUMBERS

- *(approximately 1 hr, usually) + (approximately 45 min, usually)*
- *What is the square root of (approximately 100, very likely?)*
- *(A, B)$*$(C, D)=?(E, F)*
- *f(A, B)=?(C, D)*

COMPUTATION WITH Z-NUMBERS (CONTINUED)

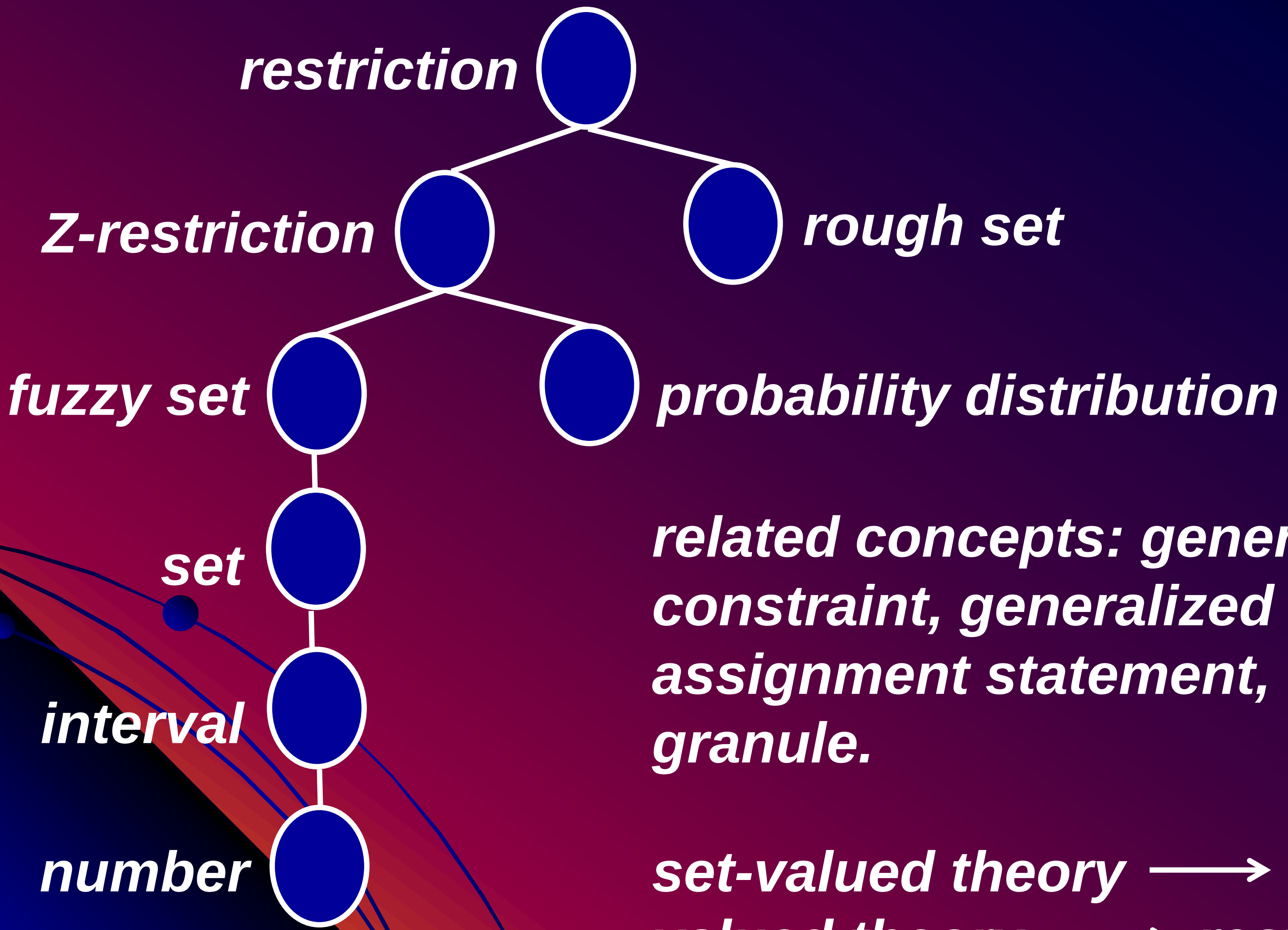
- *There are two concepts which play an essential role in computation with Z-numbers—the concept of a restriction and the concept of extension principle. These concepts are briefly discussed in the following.*

THE CONCEPT OF A RESTRICTION

THE CONCEPT OF A RESTRICTION

- ***The centerpiece of the calculus of Z-numbers is a deceptively simple concept—the concept of a restriction. The concept of a restriction has greater generality than the concept of interval, set, fuzzy set, rough set and probability distribution. An early discussion of the concept of a restriction appeared in Zadeh 1975 "Calculus of fuzzy restrictions."***

GENERALITY OF RESTRICTIONS



related concepts: generalized constraint, generalized assignment statement, granule.

set-valued theory $\longrightarrow$ fuzzy set-valued theory $\longrightarrow$ restriction-valued theory

WHAT IS A RESTRICTION?

- Informally, a restriction, R(X), on a variable, X, is an answer to a question of the form: What is the value of X?

Examples

- X=5
- X is between 3 and 7
- X is small
- It is likely that X is small
- Usually X is much larger than approximately 5.

CANONICAL FORM OF A RESTRICTION

- *The canonical form of a restriction is expressed as*

$$R(X): X \text{ isr } R,$$

where X is the restricted variable, R is the restricting relation and r is an indexical variable which defines the way R restricts X.

- *Note. In the sequel, the term restriction is sometimes applied to R.*

SINGULAR RESTRICTIONS

- There are many types of restrictions. A restriction is singular if R is a singleton. Example. X=5. A restriction is nonsingular if R is not a singleton. Nonsingularity implies uncertainty.

INDIRECT RESTRICTIONS

- *A restriction is direct if the restricted variable is X. A restriction is indirect if the restricted variable is of the form f(X). Example.*

$$R(p): \int_a^b \mu(u)p(u)\,du \text{ is likely.}$$

is an indirect restriction on p.

PRINCIPAL TYPES OF RESTRICTIONS

- ***The principal types of restrictions are: possibilistic restrictions, probabilistic restrictions and Z-restrictions.***

POSSIBILISTIC RESTRICTION (r=blank)

R(X): X is A,

where A is a fuzzy set in a space, U, with the membership function, μ_A. A plays the role of the possibility distribution of X,

$$Poss(X=u)=\mu_A(u).$$

EXAMPLE—POSSIBILISTIC RESTRICTION

X is small

↑ *restricted variable* ↑ *restricting relation (fuzzy set)*

- ***The fuzzy set small plays the role of the possibility distribution on X.***

PROBABILISTIC RESTRICTION (r=p)

R(X): X isp p,

where p is the probability density function of X,

$$Prob(u \le X \le u+du) = p(u)du.$$

EXAMPLE—PROBABILISTIC RESTRICTION

$$X \ \ isp \ \ \frac{1}{\sqrt{2\pi}} \exp(-(X-m)^2/2\sigma^2).$$

restricted variable

restricting relation (probability density function)

Z-RESTRICTION (r=z, s is suppressed)

- X is a real-valued random variable.
- A Z-restriction is expressed as

R(X): X iz Z,

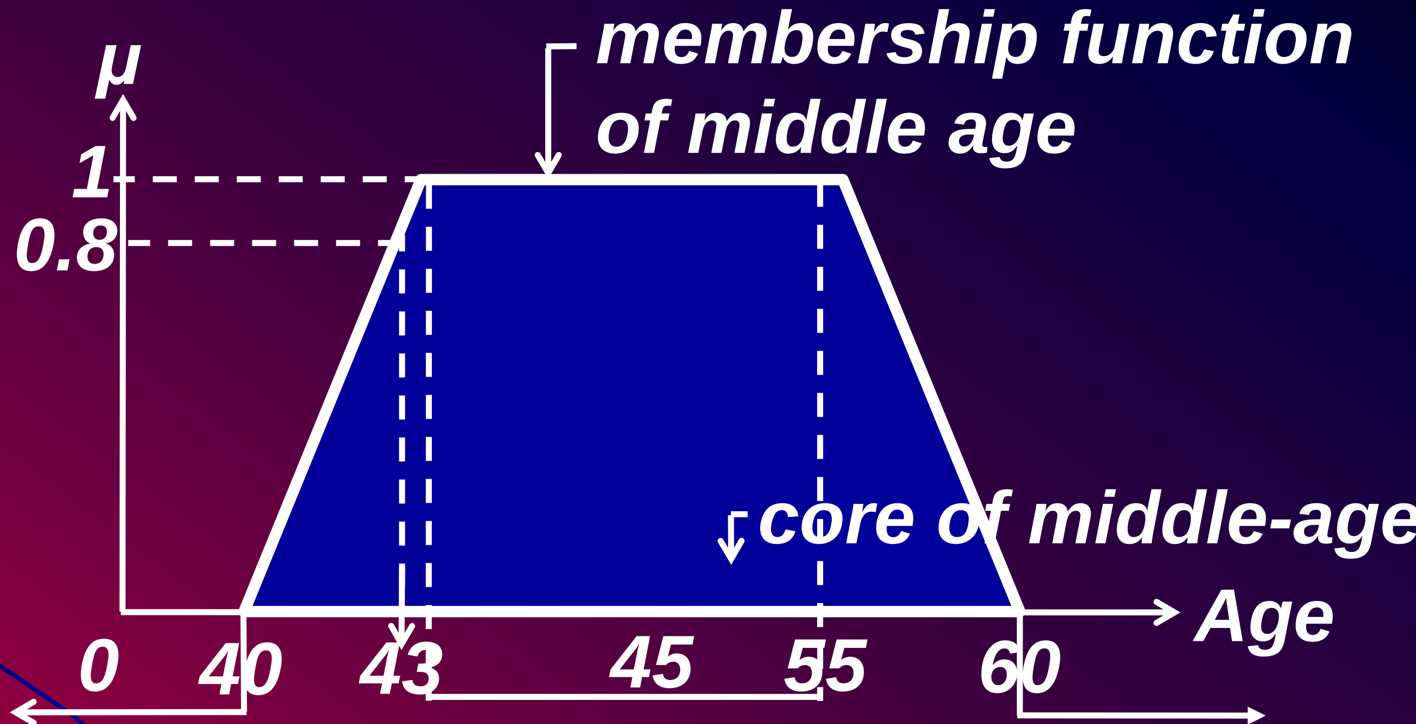
where Z is a combination of possibilistic and probabilistic restrictions defined as

Z: Prob(X is A) is B,

NATURAL LANGUAGE ≈ SYSTEM OF POSSIBILISTIC RESTRICTIONS

- ***A natural language may be viewed as a system of restrictions. In the realm of natural languages, restrictions are predominantly possibilistic. For simplicity, restrictions may be assumed to be trapezoidal.***

TRAPEZOIDAL POSSIBILISTIC RESTRICTION ON AGE



definitely not middle-age

definitely middle-age

definitely not middle-age

- *Note. Parameters are context-dependent.*

BACK TO Z-NUMBERS

- *X iz (A, B)*
- *The Z-number (A, B) is a restriction on X.*
- *X is a random variable with unknown probability density function, p. p is referred to as the underlying probability density function of X.*

THE UNDERLYING PROBABILITY DENSITY FUNCTION

- Probability measure of A may be expressed as (Zadeh 1968)

$$\int \mu_A(u)p(u)du$$

or, more simply as,

$$\mu_A \cdot p,$$

where $\cdot$ is the scalar product.

IMPORTANT OBSERVATION

- *In a Z-number, (A, B), B is an indirect possibilistic restriction on the probability measure of A. More concretely,*

$$\mu_A \cdot p \ \text{ is } \ B.$$

This expression relates p to B.

- *If X=(A, B), then (A, B) is a direct Z-restriction on X.*

NOTE

- when there is a need for placing in evidence the underlying probability density function, p, a Z-number may be expressed as

$$(A, p, B),$$

with the understanding that

$$\mu_A \cdot p \text{ is } B.$$

COMPUTATION WITH Z-NUMBERS

COMPUTATION WITH Z-NUMBERS

- Computation with Z-numbers plays an essential role in computation with Z-information. Stated in general terms, computation with Z-numbers involves evaluation of an n-ary function, f, whose arguments are Z-numbers. For simplicity, in the following, n is assumed to be 1 or 2.

SIMPLE EXAMPLES OF COMPUTATION

- *(about 1 hour, usually) + (about 2 hours, very likely)*
- *(small, usually) × (about 4, very likely)*
- *the square root of (about 30, likely)*

COMBINATION OF TWO Z-NUMBERS

- Let $X=(A_X,B_X)$ and $Y=(A_Y,B_Y)$ be Z-numbers in the space of real numbers. Let Z be a combination of X and Y, $Z=X*Y$. Example. $*=+$ More concretely,

$$Z=(A_X,B_X)*(A_Y,B_Y)$$

COMBINATION OF FUZZY NUMBERS

- Consider a generic combination, $Z=(A_Z * B_Z)$, of two fuzzy numbers $X=(A_X, B_X)$ and $Y=(A_Y, B_Y)$.

$$Z=X*Y.$$

More explicitly,

$$(A_Z, B_Z)=(A_X, B_X)*(A_Y, B_Y)$$

- The goal of computation is determination of A_Z and B_Z.

DEFINITION OF A_Z AND B_Z

$$A_Z = A_X * B_X$$

$$B_Z = \mu_{A_Z} \cdot p_Z,$$

where p_Z is the underlying probability density function of Z. More concretely,

$$p_Z = p_X \circ p_Y,$$

CONTINUED

where $\circ$ is the counterpart of $*$ for probability density functions. Simple example. When $*$=+, $\circ$=convolution.

- *Computation of p_Z would be a simple matter if we knew p_X and p_Y. What we know are possibilistic restrictions on p_X and p_Y. Specifically,*

CONTINUED

$$\mu_{A_X} \cdot p_X \text{ is } B_X$$

$$\mu_{A_Y} \cdot p_Y \text{ is } B_Y$$

At this point, computation of B_Z reduces to computation of

$$p_Z = p_X \circ p_Y,$$

with the above restrictions on p_X and p_Y.

EXTENSION PRINCIPLE

- Computation of p_Z requires the use of a basic version of the extension principle (Zadeh 1965, 1975b). Assume that Z is a function of X and Y

$$Z=f(X,Y)$$

in which X and Y are subject to possibilistic restrictions

CONTINUED

g(X) is A

h(Y) is B,

where A and B are fuzzy sets.

- *In this version of the extension principle, Z may be expressed as the solution of the variational problem*

$$\mu_Z(w)=\sup_{u,v}(\mu_A(g(u))\wedge\mu_B(h(v)))$$

CONTINUED

subject to

$$w=f(u,v)$$

Applying this version of the extension principle to the computation of p_Z, we arrive at the following definition of the combination of two Z-numbers

$$(A_X B_X)*(A_Y,B_Y)=((A_X*A_Y),\mu_{A_X*A_Y}\cdot p_Z$$

where μ_{p_Z} is given by

CONTINUED

$$\mu_{p_Z}(w) = \sup_{u,v}(\mu_{B_X}(\mu_{A_X}\cdot u)) \wedge$$

$$(\mu_{B_Y}(\mu_{A_Y}\cdot v))$$

subject to

$$w = u \circ v,$$

where u, v, and w take values in the space of probability density functions.

OBSERVATION

- *What we see is that computation of a combination of two Z-numbers is not a simple matter. Complexity of computation reflects a basic fact—construction of realistic models of complex systems carries a price.*

CONCLUDING REMARK

- ***Computation with Z-numbers is a move into uncharted territory. A variety of issues remain to be explored. One such issue is that of informativeness of results of computation. What is of relevance is that a restriction is a carrier of information. Informativeness of a restriction relates to a measure of the information which it carries. Informativeness is hard to define because it is application-dependent.***

CONTINUED

- ***In general, informativeness of a combination of Z-numbers is lessened by combination. To minimize the loss of informativeness and reduce the complexity of computations, it may be expedient to make simplifying assumptions about the underlying probability density functions. A discussion of this issue may be found in Zadeh 2011, A Note on Z-numbers.***

CONTINUED

- *This presentation focuses on the basics of the calculus of Z-numbers. No applications are discussed. Applications of Z-numbers are just beginning to get developed. A start-up, Z-Advanced Systems, is focusing on the development of applications of Z-numbers. A team of researchers headed by Professor Rafik Aliev is exploring applications of Z-numbers at the Azerbaijan State Oil Academy. Papers by other researchers are in preparation.*